The Equations Of Motion Of

Two Degree Of Freedom Systems

Systems that required two independent coordinates to describe their motion are called two defree of freedom

We begin by deriving the differential equations of motion for the two-degree-of-freedom system.

Examples

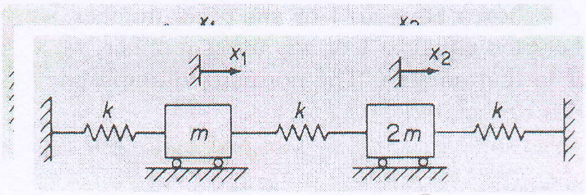


Figure (1)

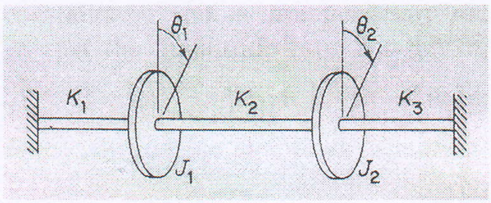


Figure (2)

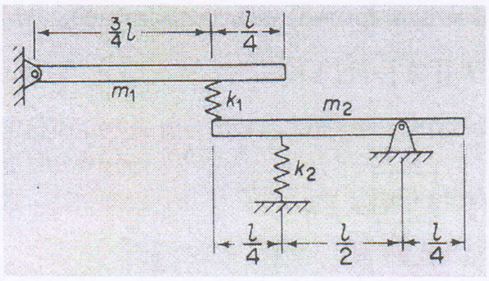


Figure (3)

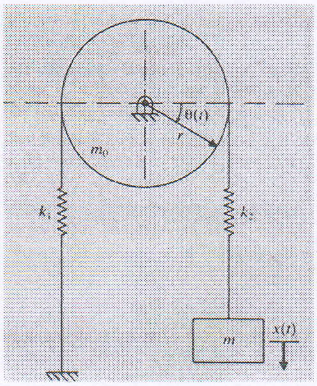


Figure (4)

Example Determine the mass matrix and stiffness matrix of the system shown in Figure (1), then find the natural frequencies and mode shapes.

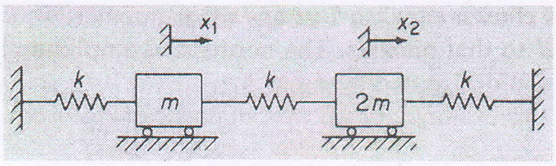
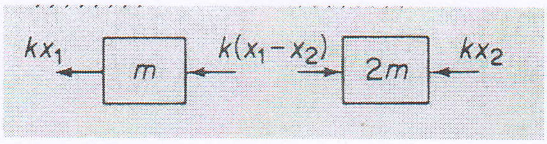


Figure (1)

Solution

Let $x_1(t) > x_2(t)$

From the free body diagram for m_1



$$\sum \mathbf{F} = m_1\ddot{x}_1 \quad \Rightarrow \quad -kx_1 - k(x_1 - x_2) = m_1\ddot{x}_1$$

$$m_1\ddot{x}_1 + 2kx_1 - kx_2 = 0 \qquad \cdots\cdots\cdots (1)$$

From the free body diagram for m_2

$$\sum \mathbf{F} = m_2\ddot{x}_2 \quad \Rightarrow \quad -kx_2 + k(x_1 - x_2) = m_2\ddot{x}_2$$

$$m_2\ddot{x}_2 + 2kx_2 - kx_1 = 0 \qquad \cdots\cdots\cdots (2)$$

Equations (1) and (2) are **equations of motion**

In solving vibration problems, matrix methods are indispensable,

Similarly, Equations (1) and (2) have the matrix form ($m_1 = m$, $m_2 = 2m$

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{M}\ddot{\mathbf{x}} + \boldsymbol{K}\mathbf{x} = \mathbf{0}$$

in which

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} = \text{mass matrix}, \qquad \boldsymbol{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} = \text{stiffness matrix}$$

To find the natural frequencies and mode shapes, let

$$x_1(t) = A_1 e^{i\omega t}, \qquad x_2(t) = A_2 e^{i\omega t}$$

$$\ddot{x}_1(t) = -A_1\omega^2 e^{i\omega t} \quad , \qquad \ddot{x}_2(t) = -A_2\omega^2 e^{i\omega t}$$

Then from the equations (1) and (2)

$$(2k - \omega^2 m)A_1 - kA_2 = 0 \qquad \cdots\cdots\cdots (3)$$

$$-kA_1 + (2k - 2\omega^2 m)A_2 = 0 \qquad \cdots\cdots\cdots (4)$$

By putting Equations (3) and (4) as a matrix form

substitution these Equations into Equation (1)

$$\begin{bmatrix} (2k - \omega^2 m) & -k \\ -k & (2k - 2\omega^2 m) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

since $\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

then
$$\begin{vmatrix} (2k-\omega^2 m) & -k \\ -k & (2k-2\omega^2 m) \end{vmatrix} = 0$$

which represents a quadratic equation in $\omega^2 = \lambda$ called the *characteristic equation*, or *frequency equation*.

$$\lambda^2 - \left(3\frac{k}{m}\right)\lambda + \frac{3}{2}\left(\frac{k}{m}\right)^2 = 0$$

The two roots λ_1 and λ_2 of this equation are the *eigenvalues* of the system

$$\lambda_1 = \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}\right)\frac{k}{m} = 0.634\frac{k}{m}$$

$$\lambda_2 = \left(\frac{3}{2} + \frac{1}{2}\sqrt{3}\right)\frac{k}{m} = 2.366\frac{k}{m}$$

and the ***natural frequencies*** of the system are

$$\omega_1 = \lambda_1^{1/2} = \sqrt{0.634\frac{k}{m}} \quad , \qquad \omega_2 = \lambda_2^{1/2} = \sqrt{2.366\frac{k}{m}}$$

From either of equations (3) or (4)

$$\frac{A_1}{A_2} = \frac{k}{2k - \omega^2 m}$$

substituting $\omega^2 = \omega_1^2 = 0.634\frac{k}{m}$

$$\left(\frac{A_1}{A_2}\right)^{(1)} = \frac{k}{2k - \omega_1^2 m} = \frac{1}{2 - 0.634} = 0.731$$

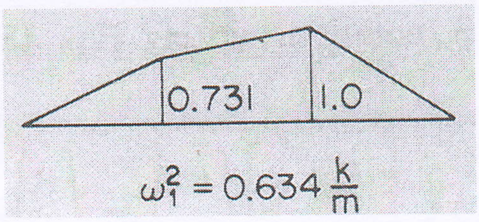
Similarly, substituting $\omega^2 = \omega_2^2 = 2.366\frac{k}{m}$

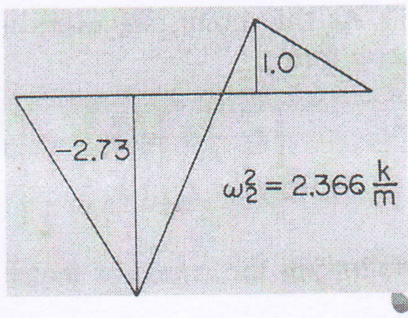
$$\left(\frac{A_1}{A_2}\right)^{(2)} = \frac{k}{2k - \omega_2^2 m} = \frac{1}{2 - 2.366} = -2.73$$

If one of the amplitudes is chosen equal to 1 or any number, we say that the amplitude ration is ***normalized*** to that number. The normalized amplitude ratio is then called the **normal mode** and is designated by $\phi_i(x)$.

The two normal modes of our example are

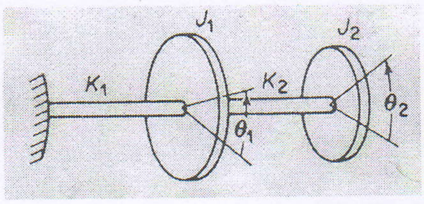
$$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \quad , \qquad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix}$$





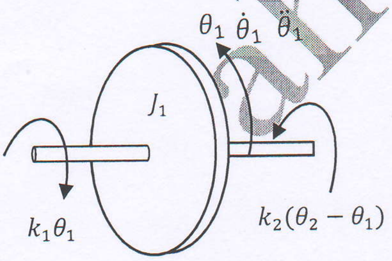
Example

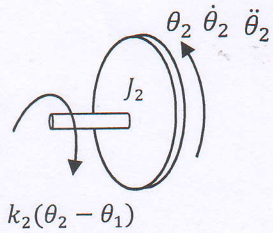
Find the mass matrix and stiffness matrix of the system shown in Figure where J_1 and J_2 are the mass moment of inertia of two disks.



Solution

Let $\theta_2 > \theta_1$





From free body diagram for J_1

$\sum M_0 = J_1\ddot{\theta}_1$ where $\ddot{\theta}_1$ is angular acceleration of J_1

$$-k_1\theta_1 + k_2(\theta_2 - \theta_1) = J_1\ddot{\theta}_1$$

$$J_1\ddot{\theta}_1 + (k_1 + k_2)\theta_1 - k_2\theta_2 = 0 \qquad \cdots\cdots\cdots\cdots (1)$$

From free body diagram for J_2

$\sum M_0 = J_2\ddot{\theta}_2$ where $\ddot{\theta}_2$ is angular acceleration of J_2

$$-k_2(\theta_2 - \theta_1) = J_2\ddot{\theta}_2$$

$$J_2\ddot{\theta}_2 + k_2\theta_2 - k_1\theta_1 = 0 \qquad \cdots\cdots\cdots\cdots (2)$$

Equations (1) and (2) are **equations of motion**, Equations (1) and (2) have the matrix form

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_1 & k_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

In solving vibration problems, matrix

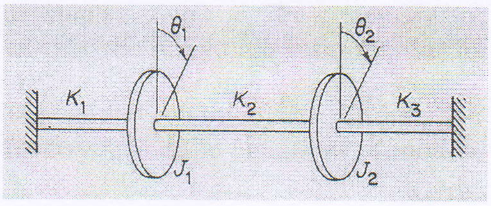
$$\boldsymbol{M}\ddot{\boldsymbol{\theta}} + \boldsymbol{K}\boldsymbol{\theta} = \boldsymbol{0}$$

where $\boldsymbol{\theta} = [\theta_1 \quad \theta_2]^T$ is two dimensional displacement vector

$$\boldsymbol{M} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_1 & k_2 \end{bmatrix}$$

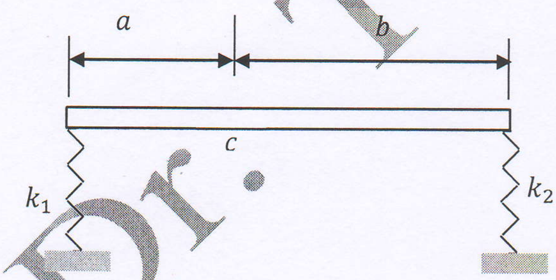
H.W

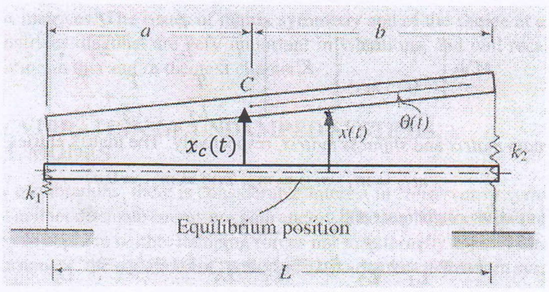
Find the mass matrix and stiffness matrix of the system shown in Figure



Example

Another two-degree-of-freedom system of interest of a slab supported on two springs as shown in Figure (simplified model of the automobile)





Solution

Since the slab has general motion (translating + rotation)